

Optimal control problems with state constraint governed by Navier-Stokes equations*

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Abstract

This work deals with the existence of optimal solution and the maximum principle for optimal control problem governed by Navier-Stokes equations with state constraint in 3-D. Strong results in 2-D also are given.

Keywords: Navier-Stokes equations; Existence; Maximum principle; State constraint.

1 Introduction

In this paper, we shall study the optimal control problem

$$(P) \quad \text{Minimize} \quad \frac{1}{2} \int_0^T \left(\int_{\Omega} |\mathcal{C}(y(t, x) - y^0(t, x))|^2 dx \right) dt + \int_0^T h(u(t)) dt;$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y + \nabla p = D_0 u(t) + f_0(t), & \text{in } \Omega \times (0, T), \\ y(0) = y_0 & \text{in } \Omega, \\ \nabla \cdot y = 0 & \text{in } \Omega \times (0, T), \\ y = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \quad (1.1)$$

$$y(t) \in K, \quad \forall t \in (0, T), \quad (1.2)$$

where K is a closed convex subset in

$$H = \{y; y \in (L^2(\Omega))^N, \nabla \cdot y = 0, y \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}. \quad (1.3)$$

Here Ω is a bounded and open subset of \mathbb{R}^N with smooth boundary $\partial\Omega$, $T > 0$ is a given constant, $\nu > 0$ is the viscosity constant, $f_0 \in L^2(0, T; (L^2(\Omega))^N)$ is a source field, $y(x, t)$ is the velocity vector, p stands for the pressure, $D_0 \in L(U; (L^2(\Omega))^N)$, and $u \in L^2(0, T; U)$, where U is a Hilbert space.

The function $h : U \rightarrow (-\infty, +\infty]$ is convex and lower semicontinuous, $y^0 \in L^2(0, T; H)$, and $\mathcal{C} \in L(V, H)$, where $V = ((H_0^1(\Omega)))^N \cap H$. Two cases of physical interest are covered by

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the cost functional of this form,

(a) $\mathcal{C} = d_1 I$: the objective is to minimize the energy of control and the difference between state function and object function;

(b) $\mathcal{C} = d_2 \nabla \times$: the objective in this case is to minimize the energy of control and regularize the smoothness of the state function.

Physically, the cost functional in case (a) means a regulation of turbulent kinetic energy, while in case (b) it means a regulation of the square of vorticity. (See [6,7,8] for a discussion on this control problem.)

Let us introduce some functional spaces and some operators to represent the Navier-Stokes equation (1.1) as infinite dimensional differential equations.

Denote by the symbol $\| \cdot \|$ the norm of the space V , which is defined by

$$\| y \|^2 = \sum_{i=1}^N \int_{\Omega} |\nabla y_i|^2 dx,$$

and by the symbol $|\cdot|$ the norm of \mathbb{R}^N and $(L^2(\Omega))^N$. We endow the space H with the norm of $(L^2(\Omega))^N$, and denote by $\langle \cdot, \cdot \rangle$ the scalar product of H , $\langle \cdot, \cdot \rangle_{(V, V')}$ the pairing between V and its dual V' with the norm $\| \cdot \|_{V'}$. Let $A \in L(V, V')$ and $b : V \times V \times V \rightarrow \mathbb{R}$ be defined by:

$$\langle Ay, z \rangle = \sum_{i=1}^N \int_{\Omega} \nabla y_i \cdot \nabla z_i dx, \quad \forall y, z \in V$$

and

$$b(y, z, w) = \sum_{i=1}^N \int_{\Omega} y_i D_i z_j w_j dx, \quad \forall y, z, w \in V$$

respectively, where $D_i = \frac{\partial}{\partial x_i}$. $D(A) = (H^2(\Omega))^N \cap V$. We define $B : V \rightarrow V'$ by

$$\langle B(y), w \rangle = b(y, y, w), \quad \forall y, w \in V$$

Let $f(t) = P f_0(t)$ and $D \in L(U, H)$ be given by $D = P D_0$, where $P : (L^2(\Omega))^N \rightarrow H$ is the projection on H . Then we may rewrite the optimal control problem (P) as:

$$(P) \quad \mathbf{Min} \quad \frac{1}{2} \int_0^T |\mathcal{C}(y(t) - y^0(t))|^2 + \int_0^T h(u(t)) dt;$$

subject to

$$\begin{cases} y'(t) + \nu Ay(t) + By(t) = Du(t) + f(t), \\ y(0) = y_0, \end{cases} \quad (1.4)$$

with

$$y(t) \in K \quad \forall t \in [0, T] \quad (1.5)$$

Since $f, Du \in L^2(0, T; H)$, $y_0 \in V$, equation (1.4) has a unique solution $y \in W^{1,2}(0, T; H) \cap L^2(0, T; D(A))$ when $N = 2$ while in the case $N = 3$, for each $u \in L^2(0, T; U)$, there exists $0 < T(u) \leq T$ such that (1.4) has a unique solution $y(\cdot; u) \in W^{1,2}(0, T^*; H) \cap L^2(0, T^*; D(A))$ for all $T^* < T(u)$. Here $T(u)$ is given by

$$T(u) = \frac{\nu}{3C_0^3 [\|y_0\|^2 + (\frac{1}{\nu}) \|f + Du\|_{L^2(0, T; H)}^2]^3} \quad (1.6)$$

where C_0 is a positive constant independent of y_0, u and ν (see [3], p.261, Th.5.10). In order to formulate the optimal control problem governed by such system in terms of strong state $y(\cdot; u)$, we observe from (1.6) that for each $L > 0$, there exists $T(L) > 0$, such that for any $T^* < T(L)$ and any $u \in L^2(0, T; U)$ with $\|Du\|_{L^2(0, T; H)} \leq L$, equation (1.4) has a unique solution $y(\cdot; u) \in W^{1,2}(0, T^*; H) \cap L^2(0, T^*; D(A))$. Therefore, the optimal control problem is well-posed in the sense of strong solutions if we consider the admissible control set as a bounded subset of $L^2(0, T; U)$. Another way to formulate the control problem is in the framework of weak solutions to equation (1.4), that is $y \in \mathcal{Y}_w = L^2(0, T; V') \cap C_w(0, T; H) \cap W^{1,1}(0, T; V')$, satisfying (see [3], p.265, Th.5.12), for each $\Psi \in V$

$$\begin{cases} \frac{d}{dt} \langle y(t), \Psi \rangle_{(V', V)} + \nu a(y, \Psi) + b(y, y, \Psi) = \langle Du + f, \Psi \rangle_{(V', V)}, \text{ a.e. } t \in (0, T) \\ y(0) = y_0. \end{cases} \quad (1.7)$$

where $C_w(0, T; H)$ is the space of weak continuous functions $y : [0, T] \rightarrow H$. It is known that there exists at least a weak solution to equation (1.4) for each $u \in L^2(0, T; U)$ (see [3], p.265, Th.5.12). We shall denote $\mathcal{P}_w = \{(y, u) \in \mathcal{Y}_w \times L^2(0, T; U); (y, u) \text{ solution to (2.4), } y(t) \in K, \forall t \in [0, T]\}$.

The main results of this work are about the existence of optimal solution and maximum principle for problem (P) in 3-D. In [5, 7, 11], some existence results are given for optimal control problems governed by Navier-Stokes equations, wherein the admissible state functions are considered as the strong solutions to Navier-Stokes equations while in the present work, we give the existence result in the framework of weak solutions to equation (1.4). In [5, 10, 11], some Pontryagin's maximum principle type results are derived for optimal control problems governed by Navier-Stokes equations. The main differences between the present work and works mentioned above are as follows. In this paper, we shall give the maximum principle for problem (P) with state constraint of pointwise type, i.e. (1.5), and it is not studied in [5, 10, 11], wherein the types of state constraint involved include type of integral, type of two point boundary and periodic type. We shall not only consider the state constraint set K as a closed convex subset of H , but also derive the maximum principle when K is a closed convex subset of V . Since the state constraint in the second case is stronger, the proof is more precise, and the corresponding result is weaker, but physically, it can be applied in some important examples in fluid mechanics which will be given in section 4. This is also one advantage of the results derived in this paper over those in the mentioned works.

The outline of this paper is as follows. In section 2, we give and prove the existence of the optimal pair for problem (P) by considering the weak admissible pair set \mathcal{P}_w . In section 3, we shall formulate the optimal control problem in terms of strong state function, which is different from that in section 2, and we get the first order necessary conditions for problem (P) with state constraint in two different cases mentioned above respectively. In section 4, we give some examples of state constraint covered by the two cases.

The following hypothesis will be in effected throughout this paper:

- (i) $K \subset H$ is a closed convex subset with nonempty interior;
- (ii) $\mathcal{C} \in L(V; H)$, $D \in L(U; H)$, $y^0 \in L^2(0, T; H \cap D(\mathcal{C}^*\mathcal{C}))$, $f \in L^2(0, T; H)$, $y_0 \in V$;
- (iii) $h : U \rightarrow (-\infty, +\infty]$ is a convex lower semicontinuous function. Moreover, there exist $\alpha > 0$ and $C \in \mathbb{R}$ such that

$$h(u) \geq \alpha |u|_U^2 + C, \quad \forall u \in U. \quad (1.8)$$

When we study problem (P) in the case that K is a closed convex subset of V , we need assumption (ii') which is assumption (ii) together with the assumption $D \in L(U; V)$.

We recall some properties of $b(y, z, w)$ here (see details in [3,9]):
 $b(y, z, w) = -b(y, w, z)$, and there exists a positive constant C , such that

$$|b(y, z, w)| \leq C \|y\|_{m_1} \|z\|_{m_2+1} \|w\|_{m_3}$$

where m_1, m_2, m_3 are positive numbers, satisfying:

$$\begin{cases} m_1 + m_2 + m_3 \geq \frac{N}{2}, & \text{if } m_i \neq \frac{N}{2}, \forall i \in \{1, 2, 3\} \\ m_1 + m_2 + m_3 > \frac{N}{2}, & \text{if } \exists i \in \{1, 2, 3\}, m_i = \frac{N}{2} \end{cases}$$

We note also the interpolation inequality:

$$\|y\|_m \leq C \|y\|_l^{1-\alpha} \|y\|_{l+1}^\alpha$$

where $\alpha = m - l \in (0, 1)$. Here $\|\cdot\|_m$ denotes the norm of the Sobolev space $H^m(\Omega)$.

We give some definition which will be used throughout this paper.

Definition 1. Given a Banach space E and its dual space E' , we denote by $BV(0, T; E')$ the space of all functionals $y : [0, T] \rightarrow E'$ with bounded variation. For each $\omega \in BV(0, T; E')$, we define the continuous functional μ_ω on $C([0, T]; E)$ by

$$\mu_\omega(z) = \int_0^T (z(t), d\omega(t))_{(E, E')}, \quad \forall z \in C([0, T]; E) \quad (1.9)$$

Here $(\cdot, \cdot)_{(E, E')}$ denotes the dual product between E and E' , and the integral takes in the Riemann-Steiljes sense. The measure μ_ω will be denoted by $d\omega$, and if we assume the space E' is reflexive, then we have the Lebesgue decomposition

$$d\omega(t) = \omega_a dt + d\omega_s(t) \quad (1.10)$$

where $\omega_a \in L^1(0, T; E')$, $\omega_a dt$ is the absolutely continuous part of measure $d\omega$, and the functional $\omega_s \in BV(0, T; E')$ is the singular part of ω . In other words, there exists a closed subset $\Theta \in [0, T]$ with the Lebesgue measure zero such that $d\omega_s = 0$, on $[0, T] \setminus \Theta$ (see [2], p.51-p.57).

Denote by $M(0, T; E')$ the dual space of $C([0, T]; E)$, i.e. the space of all bounded E' -valued measures on $[0, T]$, and notice that $\mu_\omega \in M(0, T; E')$. We denote $\mathcal{K} = \{y \in C([0, T]; E); y(t) \in K, \forall t \in [0, T]\}$, and define the normal cone to \mathcal{K} at y by

$$\mathcal{N}_{\mathcal{K}}(y) = \{\mu \in M(0, T; E'); \mu(y - x) \geq 0, \forall x \in \mathcal{K}\} \quad (1.11)$$

2 Existence results

By admissible pair we mean $(y, u) \in \mathcal{P}_w$, which satisfies equation (1.4) in the weak sense, i.e. (1.7). An optimal pair is an admissible pair which minimizes (P).

Theorem 1. *The optimal control problem (P) has at least one optimal pair (\hat{y}, \hat{u}) . In 2-D, \hat{y} is strong solution to equation (1.4).*

Proof: When $N=3$, we denote

$$F(y, u) = \frac{1}{2} \int_0^T |\mathcal{C}(y(t) - y^0(t))|^2 dt + \int_0^T h(u(t)) dt$$

$$d_1 = \inf \left\{ \frac{1}{2} \int_0^T |\mathcal{C}(y(t) - y^0(t))|^2 dt + \int_0^T h(u(t)) dt; (y, u) \in \mathcal{P}_w \right\}.$$

Then there exist $(y_n, u_n) \in \mathcal{P}_w$, such that

$$d_1 \leq F(u_n, y_n) \leq d_1 + \frac{1}{n}. \quad (2.1)$$

By (1.8) and (2.1), it follows that $\{u_n\}$ is bounded in $L^2(0, T; U)$. Hence, there exists at least a subsequence which again denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup \hat{u} \text{ weakly in } L^2(0, T; U). \quad (2.2)$$

Multiplying equation

$$\begin{cases} y_n'(t) + \nu A y_n(t) + B y_n(t) = D u_n(t) + f(t), \\ y_n(0) = y_0 \end{cases} \quad (2.3)$$

by y_n , integrating on $(0, t)$, we get that

$$|y_n(t)|^2 + \nu \int_0^t \|y_n\|^2 ds \leq C_1 + C_2 \int_0^t |y_n(s)|^2 ds,$$

and it follows by Gronwall's inequality that

$$|y_n(s)|^2 + \nu \int_0^T \|y_n\|^2 dt \leq C. \quad (2.4)$$

This yields that

$$\begin{aligned} y_n &\rightharpoonup \hat{y} \text{ weak}^* \text{ in } L^\infty(0, T; H), \text{ weakly in } L^2(0, T; V), \\ A y_n &\rightharpoonup A \hat{y} \text{ weakly in } L^2(0, T; V'). \end{aligned} \quad (2.5)$$

By the properties of the trilinear function \mathbf{b} , we have that

$$|\langle B y_n, w \rangle_{(V', V)}| \leq C |y_n|^{\frac{1}{2}} \|y_n\|^{\frac{3}{2}} \|w\|,$$

and it follows that

$$\int_0^T |B y_n|_{V'}^{\frac{4}{3}} dt \leq C \int_0^T \|y_n\|^2 dt \leq C. \quad (2.6)$$

Hence,

$$\int_0^T \left| \frac{dy_n}{dt} \right|_{V'}^{\frac{4}{3}} dt \leq C. \quad (2.7)$$

Finally, we obtain by (2.6) and (2.7) that

$$\frac{dy_n}{dt} \rightharpoonup \frac{d\hat{y}}{dt} \text{ weakly in } L^{\frac{4}{3}}(0, T; V') \quad (2.8)$$

$$B y_n \rightharpoonup \eta \text{ weakly in } L^{\frac{4}{3}}(0, T; V') \quad (2.9)$$

To show that (\hat{y}, \hat{u}) satisfies equation (1.4), it remains to show that $\eta(t) = B\hat{y}(t)$, *a.e.* in $(0, T)$. By (2.4), (2.7) and Aubin's compactness theorem (See[3], p.26, Th.1.20), we obtain that

$$y_n \rightarrow \hat{y} \text{ strongly in } L^2(0, T; H), \quad (2.10)$$

and it follows that

$$\begin{aligned} \int_0^T |\langle By_n - B\hat{y}, \psi \rangle_{(V', V)}| &\leq \int_0^T (|b(y_n - \hat{y}, y_n, \psi)| + |\hat{y}, b(y_n - \hat{y}, \psi)|) dt \\ &\rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad \forall \psi \in L^2(0, T; \mathcal{V}), \end{aligned} \quad (2.11)$$

where $\mathcal{V} = \{\psi \in C_0^\infty(\Omega); \operatorname{div} \psi = 0\}$. Hence, $\eta(t) = B\hat{y}(t)$, *a.e.* in $(0, T)$. Since h is convex and lower semicontinuous, we obtain that

$$d_1 \leq F(\hat{y}, \hat{u}) \leq \liminf_{n \rightarrow +\infty} F(u_n, y_n) \leq d_1 \quad (2.12)$$

We also have that for each $t \in [0, T]$, $\exists t_n \in (0, T)$, such that $\hat{y}(t_n) \in K$, and

$$\hat{y}(t_n) \rightarrow \hat{y}(t) \quad \text{weakly in } H. \quad (2.13)$$

Since K is a closed convex subset of H , it's weakly closed, and this yields that $\hat{y}(t) \in K, \forall t \in [0, T]$. Hence, (\hat{y}, \hat{u}) is an optimal pair for problem (P) . \sharp

Remark 1: As we stated in Section 1, when $N = 3$, if we assume that the admissible control set is a bounded subset of $L^2(0, T; U)$, then we can consider the strong solution in a local time interval $(0, T^*)$. By the similar method applied in the proof of Theorem 1, we can get the existence result, and the optimal state function $\hat{y} \in W^{1,2}(0, T^*; H) \cap L^2(0, T^*; D(A))$. Moreover, the same result follows when the state constraint set K is a closed convex subset of V .

3 The maximum principle

To get the maximum principle, we need to consider the strong solution of the Navier-Stokes equations. As we mentioned in Section 2, when $N = 3$, we need to consider the problem of such case with bounded admissible control set $\mathcal{U}_{ad} = \{u \in L^2(0, T; U); \|Du\|_{L^2(0, T; H)} \leq L\}$, and then we can consider the strong solution to Navier-Stokes equation in $(0, T^*)$, where $0 < T^* = T(L + \delta) < T(L)$. Here $\delta > 0$ is a fixed constant, and $T(L)$ is given by (2.3), i.e.

$$T(L) = \frac{\nu}{3C_0^3[\|y_0\|^2 + (\frac{2}{\nu})(\|f\|_{L^2(0, T; H)}^2 + L^2)]^3} \quad (3.1)$$

Denote $\mathcal{D}(h) = \{u \in L^2(0, T; U); \int_0^T h(u)dt < +\infty\}$. When $N = 3$, we shall assume that

$$\mathcal{D}(h) \subset \mathcal{U}_{ad}. \quad (3.2)$$

With this assumption, we can consider the strong solution in $[0, T^*]$ in 3-D without control constraint which is included in the definition of the function h inexplicitly. We give an example here to show that this assumption can be easily fulfilled. Let $h(u)$ be

$$h(u) = \begin{cases} 0, & \text{if } \|u\|_U \leq r; \\ +\infty, & \text{if } \|u\|_U > r \end{cases}$$

We see that if $\int_0^T h(u(t))dt < +\infty$, then $\|u(t)\|_U \leq r, \text{ a.e. in } [0, T]$, so $\|u\|_{L^2(0, T; U)} \leq C(r)$. Since in 2-D, the strong solution to equation (1.4) exists on arbitrary time interval $(0, T)$, such

assumption is unnecessary. We still denote the interval $[0, T^*]$ where assumption (3.2) holds by $[0, T]$.

We need also the following assumption:

(iv) There are $(\tilde{z}, \tilde{u}) \in C(0, T; H) \times L^2(0, T; U)$ solution to equation

$$\begin{cases} \tilde{z}'(t) + \nu A\tilde{z}(t) + (B'(y^*(t)))\tilde{z}(t) = B(y^*(t)) + D\tilde{u}(t) + f(t), \\ \tilde{z}(0) = y_0 \end{cases} \quad (3.3)$$

such that $\tilde{z}(t) \in \text{int}K$, for t in a dense subset of $[0, T]$.

Here $y^*(t)$ is the optimal state function for the optimal control problem (P) . Inasmuch as

$$B(y^*) \in L^2(0, T; H), |(B'(y^*)z, z)| \leq \frac{\nu}{4}\|z\|^2 + C_\nu|z|^2,$$

we know that equation (3.3) has a solution $\tilde{z} \in W^{1,2}(0, T; H) \cap L^2(0, T; D(A))$.

Theorem 2. *Let $(y^*(t), u^*(t))$ be the optimal pair for the optimal control problem (P) . Then under assumptions (i)~(iv), there are $p(t) \in L^\infty(0, T; H)$ and $\omega(t) \in BV(0, T; H)$, such that:*

$$D^*p(t) \in \partial h(u^*(t)) \quad a.e. [0, T], \quad (3.4)$$

$$p(t) = - \int_t^T U(s, t)(\mathcal{C}^*\mathcal{C}(y^*(t) - y^0(t)))ds - \int_t^T U(s, t)d\omega(s), \quad (3.5)$$

and

$$\int_0^T \langle d\omega(t), y^*(t) - x(t) \rangle \geq 0, \forall x \in \mathcal{K} \quad (3.6)$$

Here D^* , \mathcal{C}^* , $B'(y^*(t))^*$, are the adjoint operators of D , \mathcal{C} and $B'(y^*(t))$ respectively, where $B'(y)$ is the operator defined by

$$\langle B'(y)z, w \rangle = b(y, z, w) + b(z, y, w), \quad \forall z, w \in V$$

We recognize in (3.5) the mild form of the dual equation

$$\begin{cases} p'(t) = \nu Ap(t) + (B'(y^*(t))^*)p(t) + \mathcal{C}^*\mathcal{C}(y^*(t) - y^0(t)) + \mu_\omega(t), \quad a.e. \text{ in } (0, T) \\ p(T) = 0 \end{cases} \quad (3.7)$$

Theorem 3 below is the analogue of Theorem 2 under the weaker assumption :

(v) K is a closed convex subset of V , and there are $(\tilde{z}, \tilde{u}) \in C(0, T; H) \times L^2(0, T; U)$ solution to equation (3.3), such that $\tilde{z}(t) \in \text{int}_V K$, for t in a dense subset of $[0, T]$.

Here $\text{int}_V K$ is the interior of K with respect to topology of V .

Theorem 3. *Let $(y^*(t), u^*(t))$ be the solution for optimal control problem (P) . Then under assumptions (ii'), (iii), (v), there are $p(t) \in L^\infty(0, T; V')$, $\omega(t) \in BV(0, T; V')$, such that (3.4) and (3.5) hold, and (3.6) holds in the sense of*

$$\int_0^T \langle d\omega(t), y^*(t) - x(t) \rangle_{(V', V)} \geq 0, \forall x \in \mathcal{K} \quad (3.8)$$

We define first the approximating cost functions

$$F_\lambda(y, u) = \frac{1}{2} \int_0^T (|\mathcal{C}(y - y^0)|^2 + |u - u^*|_U^2) dt + \int_0^T (\varphi_\lambda(y) + h_\lambda(u)) dt. \quad (3.9)$$

where h_λ and φ_λ are the regularizations of h and φ respectively, that is

$$h_\lambda(u) = \inf\left\{\frac{\|u - v\|_U^2}{2\lambda} + h(v); v \in U\right\}, \quad \varphi_\lambda(y) = \inf\left\{\frac{|y - x|^2}{2\lambda} + \varphi(x); x \in H\right\} \quad (3.10)$$

Here φ is the characteristic function of K , which is defined by

$$\varphi(x) = \begin{cases} +\infty & \text{if } x \in H \setminus K \\ 0 & \text{if } x \in K. \end{cases}$$

The function φ_λ is convex, continuous, Gateaux differentiable, and $\partial\varphi_\lambda = \nabla\varphi_\lambda = (\partial\varphi)_\lambda$, which is single-valued (see details in [3], p.48, Th.2.9). Denote

$$\mathcal{P} = \{(y, u) \in C([0, T]; H) \times L^2(0, T; U); (y(t), u(t)) \text{ satisfy equation (1.4)}\}.$$

We prove first

Lemma 1. *There exists at least one optimal pair (y_λ, u_λ) for the optimal control problem:*

$$(P_\lambda) \quad \text{Min}\{F_\lambda(y, u); (y, u) \in \mathcal{P}\}.$$

and $(y_\lambda, u_\lambda) \rightarrow (y^*, u^*)$ strongly in $C([0, T]; H) \cap L^2(0, T; V) \times L^2(0, T; U)$. Moreover, $\{y_\lambda\}$ is bounded in $C([0, T]; V) \cap L^2(0, T; D(A))$.

Proof: The existence of the optimal pair follows by Theorem 1 and the arguments in remark 1. We shall show the convergence of the optimal pair (y_λ, u_λ) in 3-D, and it's easy to prove that the same results hold in 2-D by applying the similar method. Since

$$d_\lambda = F_\lambda(y_\lambda, u_\lambda) \leq F_\lambda(y^*, u^*) \leq F(y^*, u^*) = d, \quad (3.11)$$

we have that $\int_0^T h_\lambda(u_\lambda) dt \leq C, \forall \lambda > 0$. Since

$$h(J_\lambda^h(u_\lambda)) \leq h_\lambda(u_\lambda) \leq h(u_\lambda),$$

where $J_\lambda^h = (1 + \lambda\partial h)^{-1}$, we know that $\int_0^T h(J_\lambda^h(u_\lambda)) \leq C, \forall \lambda > 0$, and by assumption (3.2), we obtain that $\|D(J_\lambda^h(u_\lambda))\|_{L^2(0, T; H)} \leq L, \forall \lambda > 0$. Since

$$h_\lambda(u_\lambda) = \frac{\lambda}{2} \|\partial h_\lambda(u_\lambda)\|_U^2 + h(J_\lambda^h(u_\lambda)) \geq \frac{\lambda}{2} |\partial h_\lambda(u_\lambda)|^2 + \alpha \|J_\lambda^h(u_\lambda)\|_U^2 + C,$$

we have that $\{\lambda \|\partial h_\lambda(u_\lambda)\|^2\}$ is bounded in $L^1(0, T)$, and it follows that,

$$\int_0^T \|u_\lambda - J_\lambda^h(u_\lambda)\|_U^2 dt = \lambda \int_0^T \lambda \|\partial h_\lambda(u_\lambda)\|_U^2 dt \rightarrow 0 \text{ as } \lambda \rightarrow 0.$$

This implies that $\exists \lambda_0 > 0$, such that $\forall \lambda > \lambda_0$

$$\|Du_\lambda\|_{L^2(0, T; H)} \leq L + \frac{\delta}{2}. \quad (3.12)$$

Multiplying equation

$$\begin{cases} y'_\lambda(t) + \nu Ay_\lambda(t) + By_\lambda(t) = Du_\lambda(t) + f(t) \\ y_\lambda(0) = y_0 \end{cases} \quad (3.13)$$

by y_λ , integrating on $(0, t)$, it follows by Gronwall's inequality that

$$|y_\lambda(t)|^2 + \int_0^T \|y_\lambda(t)\|^2 dt \leq C, \quad \forall \lambda > 0. \quad (3.14)$$

Multiplying equation (3.13) by Ay_λ , integrating on $(0, t)$, with the inequality

$$|\langle By_\lambda, Ay_\lambda \rangle| \leq C \|y_\lambda\|^{\frac{3}{2}} |Ay_\lambda|^{\frac{3}{2}},$$

we obtain that

$$\|y_\lambda(t)\|^2 + \frac{\nu}{2} \int_0^T |Ay_\lambda(t)|^2 dt \leq C_0 \left(\|y_0\|^2 + \frac{1}{\nu} \int_0^T |f + Du_\lambda|^2 dt + \frac{1}{\nu} \int_0^t \|y_\lambda(s)\|^6 ds \right).$$

Here C_0 is the same constant as that in (3.1). It follows by Gronwall's inequality that

$$\|y_\lambda(t)\|^2 \leq \phi(t)$$

where

$$\begin{aligned} \phi(t) &= \left(\frac{\nu \phi^3(0)}{\nu - 3t\phi^3(0)} \right)^{\frac{1}{3}}, \quad \forall t \in \left(0, \frac{\nu}{3\phi^3(0)} \right) \\ \phi(0) &= C_0 \left(\|y_0\|^2 + \frac{1}{\nu} \int_0^T |f + Du_\lambda|^2 dt \right) \end{aligned}$$

By (3.12) and the definition of T , we have that

$$T < T_1 = \frac{\nu}{3C_0^3 \left(\|y_0\|^2 + \frac{2}{\nu} (|f|_{L^2(0,T;H)}^2 + (L + \frac{\delta}{2})^2) \right)^3} \leq \frac{\nu}{3\phi^3(0)}.$$

Hence

$$\|y_\lambda(t)\|^2 + \int_0^T |Ay_\lambda(t)|^2 dt \leq C(\delta), \quad \forall \lambda > \lambda_0. \quad (3.15)$$

We mention here that $C(\delta)$ is a constant dependent on δ , and we shall denote all the constants by C in the following without emphasis. By the properties of b , and (3.14), (3.15), we get

$$\|By_\lambda\|_{L^2(0,T;H)} \leq C \|Ay_\lambda\|_{L^2(0,T;H)} \leq C, \quad \forall \lambda > \lambda_0. \quad (3.16)$$

This yields that

$$\|(y_\lambda)'\|_{L^2(0,T;H)} \leq C, \quad \forall \lambda > \lambda_0. \quad (3.17)$$

Therefore, on a subsequence convergent to 0, again denoted by λ , we have

$$\begin{aligned} y_\lambda(t) &\rightarrow y_1(t) \text{ strongly in } C([0, T; H]) \cap L^2(0, T; V) \\ Ay_\lambda(t) &\rightarrow Ay_1(t) \text{ weakly in } L^2(0, T; H) \\ (y_\lambda(t))' &\rightarrow y_1'(t) \text{ weakly in } L^2(0, T; H) \\ u_\lambda(t) &\rightarrow u_1(t) \text{ weakly in } L^2(0, T; U) \end{aligned} \quad (3.18)$$

Since

$$|By_\lambda - By_1| \leq C \left(|y_\lambda - y_1|^{\frac{1}{2}} |Ay_\lambda| + \|y_\lambda - y_1\|^{\frac{1}{2}} |Ay_\lambda - Ay_1|^{\frac{1}{2}} \right),$$

by (3.15) and (3.18), we also have

$$By_\lambda(t) \rightarrow By_1(t) \text{ strongly in } L^2(0, T; H) \quad (3.19)$$

So $(y_1(t), u_1(t))$ is a solution to equation (2.1). Moreover, since

$$\varphi_\lambda(y_\lambda) = \frac{\lambda}{2} |\partial\varphi_\lambda(y_\lambda)|^2 + \varphi(J_\lambda^\varphi(y_\lambda)) \geq \frac{\lambda}{2} |\partial\varphi_\lambda(y_\lambda)|^2,$$

we know that $\{\lambda |\partial\varphi_\lambda(y_\lambda)|^2\}$ is bounded in $L^1(0, T)$, and since $\partial\varphi_\lambda(y_\lambda) = \frac{1}{\lambda}(y_\lambda - J_\lambda^\varphi(y_\lambda))$, where $J_\lambda^\varphi(y_\lambda)$ is defined by the inclusion $J_\lambda^\varphi(y_\lambda) - y_\lambda + \lambda\partial\varphi(J_\lambda^\varphi(y_\lambda)) \ni 0$, we have

$$\int_0^T |y_\lambda - J_\lambda^\varphi(y_\lambda)| dt \leq \lambda T \int_0^T \lambda |\partial\varphi_\lambda(y_\lambda)|^2 dt \rightarrow 0 \text{ as } \lambda \rightarrow 0,$$

so $y_\lambda - J_\lambda^\varphi(y_\lambda) \rightarrow 0$ a.e. $(0, T)$. Since $J_\lambda^\varphi(y_\lambda) \in K$, $\forall t \in [0, T]$, we have that $y_1(t) \in K$. $\forall t \in [0, T]$. By (3.18) we know that

$$\lim_{\lambda \rightarrow 0} J_\lambda^h(u_\lambda) = u_1 \text{ weakly in } L^2(0, T; U). \quad (3.20)$$

Since the convex function $u \rightarrow \int_0^T h(u) dt$ is lower semicontinuous, we obtain that

$$\liminf_{\lambda \rightarrow 0} \int_0^T h_\lambda(u_\lambda) \geq \int_0^T h(u_1) dt. \quad (3.21)$$

Inasmuch as

$$\liminf_{\lambda \rightarrow 0} F_\lambda(y_\lambda, u_\lambda) \leq \lim_{\lambda \rightarrow 0} F_\lambda(y^*, u^*) \leq F(y^*, u^*),$$

it follows by (3.18) and (3.21) that

$$\begin{aligned} \frac{1}{2} \int_0^T (|\mathcal{C}(y_1(t) - y^0(t))|^2 + |u_1(t) - u^*(t)|_U^2) dt + \int_0^T h(u_1(t)) dt &\leq \liminf_{\lambda \rightarrow 0} F_\lambda(y_\lambda, u_\lambda) \\ &\leq \frac{1}{2} \int_0^T (|\mathcal{C}(y^*(t) - y^0(t))|^2) dt + \int_0^T h(u^*(t)) dt \\ &\leq \frac{1}{2} \int_0^T (|\mathcal{C}(y_1(t) - y^0(t))|^2) dt + \int_0^T h(u_1(t)) dt. \end{aligned}$$

This yields that $u_1 = u^*$, $y_1 = y^*$ and $u_\lambda(t) \rightarrow u^*(t)$ strongly in $L^2(0, T; U)$.

Lemma 2. Let $z_\lambda(t)$ be the solution to the equation:

$$\begin{cases} z'_\lambda(t) + \nu A z_\lambda(t) + (B'(y_\lambda(t))) z_\lambda(t) = B(y_\lambda(t)) + D\tilde{u}(t) + f(t), \\ z_\lambda(0) = y_0. \end{cases} \quad (3.22)$$

Then $z_\lambda(t) \rightarrow \tilde{z}(t)$ strongly in $C([0, T]; H) \cap L^2(0, T; V)$, where $(\tilde{z}(t), \tilde{u}(t))$ is defined in equation (3.3), and $y_\lambda(t)$ is the optimal solution in lemma 1.

Proof: Multiplying equation (3.22) by $z_\lambda(t)$, we get

$$\frac{1}{2} \frac{d}{dt} |z_\lambda(t)|^2 + \nu \|z_\lambda(t)\|^2 + b(z_\lambda(t), y_\lambda(t), z_\lambda(t)) = b(y_\lambda(t), y_\lambda(t), z_\lambda(t)) + \langle f + D\tilde{u}, z_\lambda \rangle$$

Integrating on $(0, t)$, since

$$|b(z_\lambda, y_\lambda, z_\lambda)| \leq C \|z_\lambda\|^{\frac{3}{2}} |z_\lambda|^{\frac{1}{2}} \|y_\lambda\|, |b(y_\lambda, y_\lambda, z_\lambda)| \leq C \|y_\lambda\|^{\frac{3}{2}} |Ay_\lambda|^{\frac{1}{2}} |z_\lambda|,$$

we obtain by Young's inequality that

$$|z_\lambda(t)|^2 + \frac{\nu}{2} \int_0^t \|z_\lambda(s)\|^2 ds \leq C_1 \int_0^t |z_\lambda(s)|^2 ds + C_2.$$

It follows by Gronwall's inequality that

$$|z_\lambda(t)|^2 + \int_0^T \|z_\lambda(t)\|^2 dt \leq C. \quad (3.23)$$

Multiplying equation (3.22) by $Az_\lambda(t)$, integrating from 0 to t , we get that

$$\begin{aligned} & \frac{1}{2} \|z_\lambda(t)\|^2 - \frac{1}{2} \|y_0\|^2 + \nu \int_0^t |Az_\lambda(s)|^2 ds \\ &= \int_0^t b(y_\lambda(s), z_\lambda(s), Az_\lambda(s)) + b(z_\lambda(s), y_\lambda(s), Az_\lambda(s)) \\ & \quad + b(y_\lambda(s), y_\lambda(s), Az_\lambda(s)) ds + \langle f(s) + D\tilde{u}, z_\lambda(s) \rangle ds. \end{aligned}$$

Since

$$\begin{aligned} |b(y_\lambda, z_\lambda, Az_\lambda) + b(z_\lambda, y_\lambda, Az_\lambda)| &\leq C (\|y_\lambda\| \|z_\lambda\|^{\frac{1}{2}} |Az_\lambda|^{\frac{3}{2}} + \|z_\lambda\| \|y_\lambda\|^{\frac{1}{2}} |Ay_\lambda|^{\frac{1}{2}} |Az_\lambda|), \\ |b(y_\lambda, y_\lambda, Az_\lambda)| &\leq C \|y_\lambda\|^{\frac{3}{2}} |Ay_\lambda|^{\frac{1}{2}} |Az_\lambda|, \end{aligned}$$

we obtain that

$$\frac{1}{2} \|z_\lambda(t)\|^2 + \frac{\nu}{2} \int_0^t |Az_\lambda(s)|^2 ds \leq C_1 \int_0^t \|z_\lambda(s)\|^4 ds + C_2.$$

It follows by Gronwall's inequality and (3.23) that

$$\|z_\lambda(t)\|^2 + \int_0^T |Az_\lambda(t)|^2 dt \leq C \quad (3.24)$$

Since

$$|\langle (B'(y_\lambda(t))) z_\lambda(t), w \rangle| \leq C \left(|Az_\lambda(t)|^{\frac{1}{2}} + |Ay_\lambda(t)|^{\frac{1}{2}} \right) |w|$$

we obtain by (3.23) and (3.24) that

$$\int_0^T |(B'(y_\lambda(t))) z_\lambda(t)|^2 dt \leq C \quad (3.25)$$

By (3.23), (3.24) and (3.25), we get that

$$\int_0^T |z'_\lambda(t)|^2 dt \leq C \quad (3.26)$$

Hence

$$\begin{aligned} z_\lambda(t) &\rightarrow \bar{z}(t) \text{ strongly in } C([0, T; H]) \cap L^2(0, T; V) \\ Az_\lambda(t) &\rightarrow A\bar{z}(t) \text{ weakly in } L^2(0, T; H) \\ z'_\lambda(t) &\rightarrow \bar{z}'(t) \text{ weakly in } L^2(0, T; H). \end{aligned} \quad (3.27)$$

Since

$$\begin{aligned} &| \langle (B'(y_\lambda(t)))z_\lambda(t) - (B'(y^*(t)))\bar{z}(t), w \rangle | \\ &\leq C(|y_\lambda - y^*|^{\frac{1}{2}}|Az_\lambda| + \|z_\lambda - \bar{z}\|^{\frac{1}{2}}|A(z_\lambda - \bar{z})|^{\frac{1}{2}} \\ &\quad + |z_\lambda - \bar{z}|^{\frac{1}{2}}|Ay_\lambda| \|z_\lambda - \bar{z}\|^{\frac{1}{2}} + \|y_\lambda - y^*\|^{\frac{1}{2}}|A(y_\lambda - y^*)|^{\frac{1}{2}}|w|) \end{aligned} \quad (3.28)$$

and $y_\lambda \rightarrow y^*$, $z_\lambda \rightarrow \bar{z}$ strongly in $L^2(0, T; V) \cap C([0, T; H])$, we have also

$$B'(y_\lambda(t))z_\lambda(t) \rightarrow (B'(y^*(t)))\bar{z}(t) \text{ strongly in } L^2(0, T; H) \quad (3.29)$$

With above inequalities, passing λ to 0 in equation (3.22), we find that $\bar{z}(t)$ satisfies the equation (3.3), and by the uniqueness of the solution, $\bar{z}(t) = \tilde{z}(t)$. \dagger

We shall denote by $U(t, s)$ and $U_\lambda(s, t)$, $0 \leq s \leq t \leq T$ the evolution operators generated by $\nu A + (B'(y^*(t)))^*$ and $\nu A + (B'(y_\lambda(t)))^*$ respectively, which are given by $U(t, s)\xi = \psi(t)$ and $U_\lambda(s, t)x = \psi_\lambda(t)$ for $0 \leq s \leq t \leq T$, where $\psi(t)$ and $\psi_\lambda(t)$ are the solutions to

$$\begin{cases} \psi'(t) = \nu A\psi(t) + (B'(y^*(t)))^*\psi(t), \\ \psi(s) = \xi \end{cases} \quad (3.30)$$

and

$$\begin{cases} \psi'_\lambda(t) = \nu A\psi_\lambda(t) + (B'(y_\lambda(t)))^*\psi_\lambda(t), \\ \psi_\lambda(s) = \xi \end{cases} \quad (3.31)$$

respectively. It is well known and easily seen that such evolution operators exist. Denote by $U^*(t, s)$, $U_\lambda^*(s, t)$ the respective adjoint operators of $U(t, s)$ and $U_\lambda(s, t)$, which are generated by $\nu A + B'(y^*(t))$ and $\nu A + B'(y_\lambda(t))$ respectively. By the similar method applied in lemma 2, we can obtain that

$$\begin{cases} \|U_\lambda^*(t, s)\|_{L(H, H)} \leq C \\ U_\lambda^*(s, t)\xi \rightarrow U^*(s, t)\xi \text{ in } C([0, T; H]), \forall \xi \in H \end{cases} \quad (3.32)$$

Proof of Theorem 2:

step 1:(first order necessary condition for approximate problem) Since (y_λ, u_λ) minimize the functional $F_\lambda(y, u)$, we know that

$$\lim_{\rho \rightarrow 0} \frac{F_\lambda(u_\lambda + \rho u) - F_\lambda(u_\lambda)}{\rho} = 0, \quad \forall u \in U,$$

and this yields

$$\langle \mathcal{E}^* \mathcal{C}(y_\lambda - y^0), w_\lambda \rangle + \langle \nabla h_\lambda(u_\lambda) + u_\lambda - u^*, u \rangle_U + \langle \partial \varphi_\lambda(y_\lambda), w_\lambda \rangle = 0, \quad (3.33)$$

where $w_\lambda = \lim_{\rho \rightarrow 0} \frac{y_\lambda^\rho - y_\lambda}{\rho}$, $(y_\lambda^\rho, u_\lambda + \rho u) \in \mathcal{P}$, and $w_\lambda(t)$ is the solution to the equation

$$w'_\lambda(t) + \nu A w_\lambda(t) + B'(y_\lambda(t))w_\lambda(t) = Du, \quad w_\lambda(0) = 0. \quad (3.34)$$

Let $p_\lambda(t)$ be the solution to the backward dual equation

$$\begin{cases} p'_\lambda(t) = \nu A p_\lambda(t) + (B'(y_\lambda(t))^*) p_\lambda(t) + \mathcal{C}^* \mathcal{C}(y_\lambda(t) - y^0(t)) + \partial \varphi_\lambda(y_\lambda(t)) \\ p_\lambda(T) = 0 \end{cases} \quad (3.35)$$

By (3.33), (3.34) and (3.35), we get by calculation that

$$\langle p'_\lambda(t), w_\lambda(t) \rangle + \langle -A p_\lambda(t) - (B'(y_\lambda(t))^*) p_\lambda(t), w_\lambda(t) \rangle + \langle u_\lambda - u^*, u \rangle_U = 0.$$

Hence

$$\langle -D^* p_\lambda(t) + \nabla h_\lambda(u_\lambda) + u_\lambda - u^*, u \rangle_U = 0, \quad \forall u \in U$$

Finally, we obtain that

$$D^* p_\lambda(t) = \nabla h_\lambda(u_\lambda) + u_\lambda - u^*(t), \quad a.e. \ t \in [0, T] \quad (3.36)$$

step2: (pass $\partial \varphi_\lambda(y_\lambda), p_\lambda, \partial h_\lambda(u_\lambda)$ to limit) By assumption (iv) and lemma 2, we know that $\exists \rho > 0, \lambda_1 > 0$ s.t. $z_\lambda(t) + \rho h \in K$, for t in a dense subset of $[0, T]$, $\forall |h| = 1, \forall \lambda > \lambda_1$. For λ fixed, $z_\lambda(t)$ is continuous in $[0, T]$, so there exists a partition of $[0, T]$,

$$0 = t_1 < t_2 < \dots < t_{N-1} < t_N = T$$

such that $|z_\lambda(t_i) - z_\lambda(t_{i-1})| < \frac{\rho}{2}$, $z_\lambda(t_i) + \rho h \in K, \forall 1 \leq i \leq N$. Moreover, since

$$\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \partial \varphi_\lambda(y_\lambda(t)), y_\lambda(t) - z_\lambda(t_i) - \rho h \rangle dt \geq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \varphi_\lambda(y_\lambda(t)) - \varphi_\lambda(z_\lambda(t_i) + \rho h) dt \geq 0,$$

we get that

$$\begin{aligned} \rho \int_0^T |\partial \varphi_\lambda(y_\lambda)| dt &\leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \partial \varphi_\lambda(y_\lambda(t)), y_\lambda(t) - z_\lambda(t_i) \rangle dt \\ &= \int_0^T \langle \partial \varphi_\lambda(y_\lambda(t)), y_\lambda(t) - z_\lambda(t) \rangle dt + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \partial \varphi_\lambda(y_\lambda(t)), z_\lambda(t) - z_\lambda(t_i) \rangle dt, \end{aligned}$$

and it follows that

$$\begin{aligned} \frac{\rho}{2} \int_0^T |\partial \varphi_\lambda(y_\lambda)| dt &\leq \int_0^T \langle \partial \varphi_\lambda(y_\lambda(t)), y_\lambda(t) - z_\lambda(t) \rangle dt \\ &= \int_0^T \langle p'_\lambda(t) - \nu A p_\lambda(t) - (B'(y_\lambda(t))^*) p_\lambda(t) - \mathcal{C}^* \mathcal{C}(y_\lambda(t) - y^0(t)), y_\lambda(t) - z_\lambda(t) \rangle dt \\ &= \int_0^T \langle p_\lambda, -y'_\lambda - \nu A y_\lambda - (B'(y_\lambda)) y_\lambda + z'_\lambda + \nu A z_\lambda + (B'(y_\lambda)) z_\lambda \rangle + \langle \mathcal{C}^* \mathcal{C}(y_\lambda - y^0), y_\lambda - z_\lambda \rangle dt \\ &= \int_0^T \langle p_\lambda, D\tilde{u} - D u_\lambda \rangle - \langle \mathcal{C}^* \mathcal{C}(y_\lambda - y^0), y_\lambda - z_\lambda \rangle dt \\ &= \int_0^T (\langle \nabla h_\lambda(u_\lambda) + u_\lambda(t) - u^*, \tilde{u}(t) - u_\lambda(t) \rangle_U - \langle \mathcal{C}^* \mathcal{C}(y_\lambda - y^0), y_\lambda - z_\lambda \rangle) dt \end{aligned}$$

$$\leq \int_0^T [h_\lambda(u_\lambda) - h_\lambda(\tilde{u})]dt + C \leq C. \quad (3.37)$$

We set $\omega_\lambda(t) = \int_0^t \partial\varphi_\lambda(y_\lambda(s))ds$, $t \in [0, T]$, then by (3.37) and the Helly theorem (see [2], p.58, Th.3.5), we know that there exists a function $\omega \in BV([0, T]; H)$, and a sequence convergent to 0, again denoted by λ , such that

$$\omega_\lambda(t) \rightarrow \omega(t) \text{ weakly in } H \text{ for every } t \in [0, T] \quad (3.38)$$

and

$$\int_t^T \langle \partial\varphi_\lambda(y_\lambda(s)), x(s) \rangle ds \rightarrow \int_t^T \langle d\omega(s), x(s) \rangle, \quad \forall x \in C([t, T]; H), \forall t \in [0, T]. \quad (3.39)$$

Multiplying equation (3.35) by $\text{sign} p_\lambda(t) = \frac{p_\lambda(t)}{|p_\lambda(t)|}$, we get that

$$\frac{d}{dt}|p_\lambda(t)| = \frac{\nu \|p_\lambda\|^2}{|p_\lambda|} + \frac{b(p_\lambda, y_\lambda, p_\lambda)}{|p_\lambda|} + \frac{\langle \mathcal{C}^* \mathcal{C}(y_\lambda - y^0), p_\lambda \rangle}{|p_\lambda|} + \frac{\langle \partial\varphi_\lambda(y_\lambda), p_\lambda \rangle}{|p_\lambda|}.$$

Since $|b(p_\lambda(t), y_\lambda(t), p_\lambda(t))| \leq C|p_\lambda(t)|^{\frac{1}{2}}\|p_\lambda(t)\|^{\frac{3}{2}}\|y_\lambda(t)\|$, we get by Young's inequality that

$$\frac{b(p_\lambda(t), y_\lambda(t), p_\lambda(t))}{|p_\lambda(t)|} \leq C \frac{\|p_\lambda(t)\|^{\frac{3}{2}}}{|p_\lambda(t)|^{\frac{3}{4}}} |p_\lambda(t)|^{\frac{1}{4}} \leq \frac{\nu}{2} \frac{\|p_\lambda(t)\|^2}{|p_\lambda(t)|} + C|p_\lambda(t)|.$$

Integrating on (t, T) , we obtain by Young's inequality that

$$|p_\lambda(t)| + \frac{\nu}{2} \int_t^T \|p_\lambda(s)\| ds \leq C_1 + C_2 \int_t^T |p_\lambda(s)| ds.$$

It follows by Gronwall's inequality that

$$\|p_\lambda(t)\|_{L^\infty(0, T; H)} \leq C, \quad (3.40)$$

and so by Alaoglu's theorem,

$$p_\lambda(t) \rightarrow p(t) \quad w^* - L^\infty(0, T; H). \quad (3.41)$$

By (3.32), we infer that

$$\int_t^T U_\lambda(s, t) \mathcal{C}^* \mathcal{C}(y_\lambda(s) - y^0(s)) ds \rightarrow \int_t^T U(s, t) \mathcal{C}^* \mathcal{C}(y^*(s) - y^0(s)) ds \text{ weakly in } H, \quad (3.42)$$

and by (3.38), we have

$$\int_t^T U_\lambda(s, t) \partial\varphi_\lambda(y_\lambda(s)) ds \rightarrow \int_t^T U(s, t) d\omega(s) \text{ weakly in } H \quad (3.43)$$

Finally, by (3.41), (3.42) and (3.43), we obtain that

$$p(t) = - \int_t^T U(s, t) (\mathcal{C}^* \mathcal{C}(y^*(s) - y^0(s))) ds - \int_t^T U(s, t) d\omega(s).$$

This means that $p(t)$ satisfies equation (3.5). Since

$$\int_0^T \langle \partial \varphi_\lambda(y_\lambda(t)), y_\lambda(t) - x(t) \rangle dt \geq \varphi_\lambda(y_\lambda(t)) - \varphi_\lambda(x(t)) \geq 0, \forall x \in \mathcal{K},$$

by (3.39), we can pass λ to 0 to get

$$\int_0^T \langle d\omega(t), y^*(t) - x(t) \rangle \geq 0,$$

so (3.6) holds. To complete the proof, it remains to proof (3.4). By (3.36) and the definition of ∂h_λ , we have that

$$\int_0^T (h_\lambda(u_\lambda) - h_\lambda(v)) dt \leq \int_0^T \langle D^* p_\lambda - (u_\lambda - u^*), u_\lambda - v \rangle_U dt, \forall v \in L^2(0, T; U). \quad (3.44)$$

Remembering that $h_\lambda(u) \leq h(u)$, we obtain by (3.21) and (3.44) that

$$\int_0^T (h(u^*) - h(v)) dt \leq \int_0^T \langle D^* p, u^* - v \rangle_U dt, \forall v \in L^2(0, T; U). \quad (3.45)$$

This implies the pointwise inequality:

$$\langle D^* p, u^* - \tilde{v} \rangle_U \geq h(u^*) - h(\tilde{v}), a.e. \text{ in } (0, T).$$

This shows that $D^* p(t) \in \partial h(u^*(t)), a.e. \text{ in } (0, T) \#$

Proof of Theorem 3: Step 1 is the same as the proof of theorem 2. To pass $p_\lambda(t), \partial h_\lambda(u_\lambda(t))$ and $\partial \varphi_\lambda(y_\lambda(t))$ to limit, we need to prove the following lemma first:

Lemma 3. $z_\lambda(t) \rightarrow \tilde{z}(t)$ strongly in $C([0, T]; V)$, where $z_\lambda(t)$ and $\tilde{z}(t)$ are the solutions to equation (3.22) and equation (3.3) respectively.

Proof: We have

$$(z_\lambda(t) - \tilde{z}(t))' + \nu A(z_\lambda'(t) - \tilde{z}(t)) + (B'(y_\lambda(t)))z_\lambda(t) - (B'(y^*(t)))\tilde{z}(t) = B(y_\lambda(t)) - B(y^*(t)) \quad (3.46)$$

Multiplying equation (3.46) by $A(z_\lambda(t) - \tilde{z}(t))$, integrating on $(0, t)$, we get that

$$\begin{aligned} & \frac{1}{2} \|z_\lambda(t) - \tilde{z}(t)\|^2 + \nu \int_0^t |A(z_\lambda(s) - \tilde{z}(s))|^2 ds \\ &= - \int_0^t b(y_\lambda(s) - y^*(s), z_\lambda(s), Az_\lambda(s) - A\tilde{z}(s)) - \int_0^t b(y^*(s), z_\lambda(s) - \tilde{z}(s), Az_\lambda(s) - A\tilde{z}(s)) ds \\ & \quad - b(z_\lambda(s) - \tilde{z}(s), y_\lambda(s), Az_\lambda(s) - A\tilde{z}(s)) - b(\tilde{z}(s), y_\lambda(s) - y^*(s), Az_\lambda(s) - A\tilde{z}(s)) ds \\ & \quad + \int_0^t \langle B(y_\lambda(s)) - B(y^*(s)), Az_\lambda(s) - A\tilde{z}(s) \rangle ds \\ & \leq C \int_0^t \left(|y_\lambda - y^*|^{\frac{1}{2}} \|Az_\lambda\| |A(z_\lambda - \tilde{z})| + \|z_\lambda - \tilde{z}\|^{\frac{1}{2}} |Az_\lambda - A\tilde{z}|^{\frac{3}{2}} \right) ds \\ & + C \int_0^t \left(|z_\lambda - \tilde{z}|^{\frac{1}{2}} \|z_\lambda - \tilde{z}\|^{\frac{1}{2}} |Az_\lambda - A\tilde{z}| |Ay_\lambda| + \|y_\lambda - y^*\|^{\frac{1}{2}} |Ay_\lambda - Ay^*|^{\frac{1}{2}} |A(z_\lambda - \tilde{z})| \right) ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \langle B(y_\lambda(s)) - B(y^*(s)), Az_\lambda(s) - A\tilde{z}(s) \rangle ds \\
& \leq C_1 \int_0^t \|z_\lambda - \tilde{z}\|^2 ds + C_2 \left(\int_0^t |By_\lambda - By^*|^2 ds + \|y_\lambda - y^*\|_{C([0,T];V)} + \|z_\lambda - \tilde{z}\|_{C([0,T;H])} \right) \\
& \quad + \frac{\nu}{2} \int_0^t |Az_\lambda(s) - A\tilde{z}(s)|^2 ds
\end{aligned}$$

Denote $C_2 \left(\int_0^t |B(y_\lambda(s)) - B(y^*(s))|^2 ds + \|y_\lambda - y^*\|_{C([0,T];V)} + \|z_\lambda - \tilde{z}\|_{C([0,T;H])} \right)$ by ε_λ . By the latter inequality and Gronwall's inequality, it follows that

$$\sup_{t \in [0,T]} \|z_\lambda(t) - \tilde{z}(t)\|^2 \leq \varepsilon_\lambda e^{C_1 T} \rightarrow 0 \text{ as } \lambda \rightarrow 0 \quad (3.47)$$

Indeed, since $B(y_\lambda(t)) \rightarrow B(y^*(t))$ strongly in $L^2(0, T; H)$, and $z_\lambda \rightarrow \tilde{z}$ strongly in $C([0, T]; H)$, it suffices to prove that $y_\lambda \rightarrow y^*$ strongly in $C([0, T]; V)$. We have

$$\begin{cases} (y_\lambda(t) - y^*(t))' + \nu A(y_\lambda(t) - y^*(t)) + By_\lambda(t) - By^*(t) = Du_\lambda(t) - Du^*(t), \\ y_\lambda(0) - y^*(0) = 0 \end{cases} \quad (3.48)$$

Multiplying equation (3.48) by $Ay_\lambda(t) - Ay^*(t)$, integrating on $(0, t)$. It follows that

$$\begin{aligned}
& \frac{1}{2} \|y_\lambda(t) - y^*(t)\|^2 + \nu \int_0^t |Ay_\lambda(s) - Ay^*(s)|^2 ds \\
& \leq C_1 \int_0^t \|y_\lambda(t) - y^*(t)\|^2 ds + C_2 \int_0^t (|Du_\lambda(s) - Du^*(s)|^2 + \|y_\lambda - y^*\|_{C([0,T;H])}) ds \\
& \quad + \frac{\nu}{2} \int_0^t |Ay_\lambda(s) - Ay^*(s)|^2 ds.
\end{aligned}$$

Since $u_\lambda(t) \rightarrow u^*(t)$ strongly in $L^2(0, T; H)$, $y_\lambda \rightarrow y^*$ strongly in $C([0, T]; H)$, we obtain by Gronwall's inequality that

$$\|y_\lambda(t) - y^*(t)\|^2 \leq C \left(\|Du_\lambda(t) - Du^*(t)\|_{L^2(0,T;H)}^2 + \|y_\lambda - y^*\|_{C([0,T;H])} \right) dt e^{C_1 T} \rightarrow 0 \text{ as } \lambda \rightarrow 0.$$

This shows that

$$y_\lambda \rightarrow y^* \text{ strongly in } C([0, T]; V). \quad (3.49)$$

So (3.47) holds. We complete the proof of Lemma 3.

Applying the similar method, we can get the following result

$$\begin{cases} \|U_\lambda^*(t, s)\|_{L(V,V)} \leq C \\ U_\lambda^*(s, t)\xi \rightarrow U^*(s, t)\xi \text{ in } C([0, T]; V), \forall \xi \in V \end{cases} \quad (3.50)$$

Now we come back to pass $p_\lambda(t)$, $\partial h_\lambda(u_\lambda(t))$ and $\partial \varphi_\lambda(y_\lambda(t))$ to limit.

By assumption (v) and Lemma 3, we know that $\exists \rho > 0, \lambda_0 > 0$ s.t. $z_\lambda(t) + \rho h \in K$, for t in a dense subset of $[0, T]$, $\forall \lambda < \lambda_0, \forall \|h\| = 1$. By the similar arguments to the proof of Theorem 2, we can get that $\{\partial \varphi_\lambda(y_\lambda)\}$ is bounded in $L^1(0, T; V')$. We denote $\omega_\lambda(t) = \int_0^t \partial \varphi_\lambda(y_\lambda(s)) ds$, $t \in [0, T]$, then we know by the Helly theorem that there exist a functional $\omega \in BV([0, T]; V')$, and a sequence convergent to 0, again denoted by λ , such that

$$\omega_\lambda(t) \rightarrow \omega(t) \text{ weakly in } V' \text{ for every } t \in [0, T] \quad (3.51)$$

and

$$\int_t^T (\partial\varphi_\lambda(y_\lambda(s)), x(s))_{(V', V)} ds \rightarrow \int_t^T (d\omega(s), x(s))_{(V', V)}, \quad \forall x \in C([t, T]; V), \forall t \in [0, T]. \quad (3.52)$$

Multiply equation (3.35) by $\frac{A^{-1}p_\lambda(t)}{\|p_\lambda(t)\|_{V'}}$ in the sense of the dual product between V' and V , denote $q_\lambda(t) = A^{-1}p_\lambda(t)$, we have

$$\begin{aligned} \frac{d}{dt} \|p_\lambda(t)\|_{V'} &= \frac{\nu |p_\lambda(t)|^2}{\|p_\lambda(t)\|_{V'}} + \frac{b(q_\lambda, y_\lambda, p_\lambda) + b(y_\lambda, q_\lambda, p_\lambda)}{\|p_\lambda(t)\|_{V'}} \\ &\quad + \frac{\langle \mathcal{C}^* \mathcal{C}(y^*(t) - y^0(t)), q_\lambda(t) \rangle}{\|p_\lambda(t)\|_{V'}} + \frac{\langle \partial\varphi_\lambda(y_\lambda), q_\lambda(t) \rangle}{\|p_\lambda(t)\|_{V'}}. \end{aligned}$$

Since

$$\begin{aligned} \frac{b(q_\lambda, y_\lambda, p_\lambda)}{\|p_\lambda\|_{V'}} &\leq C \frac{\|q_\lambda\|_{\frac{3}{2}+\varepsilon} |p_\lambda(t)|}{\|p_\lambda(t)\|_{V'}} \leq C \|p_\lambda\|_{V'}^{\frac{1-2\varepsilon}{4}} \frac{|p_\lambda|^{\frac{3+2\varepsilon}{2}}}{\|p_\lambda(t)\|_{V'}^{\frac{3+2\varepsilon}{4}}}, \quad 0 < \varepsilon < \frac{1}{2}, \\ \frac{b(y_\lambda, q_\lambda, p_\lambda)}{\|p_\lambda\|_{V'}} &\leq C \|p_\lambda\|_{V'}^{\frac{1}{4}} \frac{|p_\lambda|^{\frac{3}{2}}}{\|p_\lambda(t)\|_{V'}^{\frac{3}{4}}}, \end{aligned}$$

we obtain by Young's inequality that

$$\|p_\lambda(t)\|_{V'} + \nu \int_t^T \frac{|p_\lambda(s)|^2}{\|p_\lambda(s)\|_{V'}} ds \leq C_1 + C_2 \int_t^T \|p_\lambda(s)\|_{V'} ds + \frac{\nu}{2} \int_t^T \frac{|p_\lambda(s)|^2}{\|p_\lambda(s)\|_{V'}} ds.$$

It follows by Gronwall's inequality that

$$\|p_\lambda(t)\|_{L^\infty(0, T; V')} \leq C, \quad (3.53)$$

and so by Alaoglu's theorem,

$$p_\lambda(t) \rightarrow p(t) \quad w^* - L^\infty(0, T; V') \quad (3.54)$$

By (3.18) and (3.50), we have

$$\int_t^T U_\lambda(s, t) \mathcal{C}^* \mathcal{C}(y_\lambda(s) - y^0(s)) ds \rightarrow \int_t^T U(s, t) \mathcal{C}^* \mathcal{C}(y(s) - y^0(s)) ds \text{ weakly in } V'. \quad (3.55)$$

By (3.51), we get

$$\int_t^T U_\lambda(s, t) \partial\varphi_\lambda(y_\lambda(s)) ds \rightarrow \int_t^T U(s, t) d\omega(s) \text{ weakly in } V'. \quad (3.56)$$

Finally, we obtain by (3.54), (3.55) and (3.56) that

$$p(t) = - \int_t^T U(s, t) (\mathcal{C}^* \mathcal{C}(y^*(t) - y^0(t))) ds - \int_t^T U(s, t) d\omega(s).$$

This means that $p(t)$ satisfies equation (3.5). Moreover, (3.4) and (3.8) also hold by applying the same arguments as that in the proof of theorem 1. \sharp

We shall consider the reflexive Banach space E as H or V , and denote by (\cdot, \cdot) the dual product between E and its dual of E (When $E = H$, it is the scalar product in H), by $\|\cdot\|$ the norm of E . Under the hypothesis of Theorem 2 or the hypothesis of Theorem 3, We give a corollary here:

Corollary 1. *Let the pair (y^*, u^*) be the optimal pair in problem (P), then there exist $\omega(t) \in BV([0, T]; E')$ and p satisfying along with y^*, u^* , equations (3.4), (3.5), (3.6) (or (3.8)) and*

$$\omega_a(t) \in N_K(y^*(t)), \text{ a.e. } t \in (0, T) \quad (3.57)$$

$$d\omega_s \in \mathcal{N}_K(y^*) \quad (3.58)$$

Here $\omega_a(t)$ is the weak derivative of $\omega(t)$, and $d\omega_s$ is the singular part of measure $d\omega$. $N_K(y^*(t))$ is the normal cone to K at $y^*(t)$, and $\mathcal{N}_K(y^*)$ is the normal cone to K at y^* which is precised in definition 1.

Proof: Let t_0 be arbitrary but fixed in $(0, T)$. For $y \in K$ and $\varepsilon > 0$, define the function y_ε

$$y_\varepsilon(t) = \begin{cases} y^*(t) & \text{for } |t - t_0| \geq \varepsilon \\ (1 - \varepsilon^{-1}(t_0 - t))y + \varepsilon^{-1}(t_0 - t)y^*(t_0 - \varepsilon) & \text{for } t \in [t_0 - \varepsilon, t_0] \\ (1 - \varepsilon^{-1}(t - t_0))y + \varepsilon^{-1}(t - t_0)y^*(t_0 + \varepsilon) & \text{for } t \in [t_0, t_0 + \varepsilon]. \end{cases}$$

Obviously y_ε is continuous from $[0, T]$ to E and $y_\varepsilon(t) \in K, \forall t \in [0, T]$. By (4.6)(or (4.8)), we have

$$\int_0^T (\dot{\omega}(t), y^*(t) - y_\varepsilon(t))dt + \int_0^T (d\omega_s, y^* - y_\varepsilon) \geq 0. \quad (3.59)$$

We set $\rho_\varepsilon(t) = \varepsilon^{-1}(y^*(t) - y_\varepsilon(t))$. If t_0 happens to be a Lebesgue point for the function ω_a , then by an elementary calculation involving the definition of y_ε , we get

$$\lim_{\varepsilon \rightarrow 0} \int_0^T (\omega_a(t), \rho_\varepsilon(t))dt = (\omega_a(t_0), y^*(t_0) - y). \quad (3.60)$$

Inasmuch as $y^*(t) - y_\varepsilon(t) = 0$ outside $[t_0 - \varepsilon, t_0 + \varepsilon]$, we have

$$\int_0^T (d\omega_s, y^* - y_\varepsilon) = \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} (d\omega_s, y^* - y_\varepsilon).$$

On the other hand, for each $\eta > 0$, there exists $\{y_{i\eta}^*\}_{i=1}^N \subset E$, and $\alpha_{i\eta} \in C([0, T])$ such that

$$\|y^*(t) - \sum_{i=1}^N y_{i\eta}^* \alpha_{i\eta}(t)\| \leq \eta \text{ for } t \in [0, T]$$

We set $z_\eta(t) = y^*(t) - \sum_{i=1}^N y_{i\eta}^* \alpha_{i\eta}(t)$, then we have

$$\left| \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} (d\omega_s, z_\eta) \right| \leq (V_s(t_0 + \varepsilon) - V_s(t_0 - \varepsilon)) \cdot \sup\{\|z_\eta\|; |t - t_0| \leq \varepsilon\}$$

where $V_s(t)$ is the variation of ω_s on the interval $[0, t]$. Since V_s is a.e. differentiable on $(0, T)$, we may assume that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} (d\omega_s, z_\eta) \leq C\eta, \quad (3.61)$$

where C is independent of η . Now, we have

$$\begin{aligned} \left| \int_{t_0-\varepsilon}^{t_0+\varepsilon} (d\omega_s, \sum_{i=1}^N y_{i\eta}^* \alpha_{i\eta}) \right| &\leq \sum_{i=1}^N \left| \int_{t_0-\varepsilon}^{t_0+\varepsilon} \alpha_{i\eta} d(\omega_s, y_{i\eta}^*) \right| \\ &\leq \sum_{i=1}^N (V_{i\eta}(t_0 + \varepsilon) - V_{i\eta}(t_0 - \varepsilon)) \gamma_{i\eta}, \end{aligned}$$

where $V_{i\eta}(t)$ is the variation of $(\omega_s, y_{i\eta}^*)$ on interval $[0, t]$ and $\gamma_{i\eta} = \sup |\alpha_{i\eta}(t)|$. Since the weak derivative of ω_s is zero a.e. on $(0, T)$. We may infer that:

$$\frac{d}{dt} V_{i\eta}(t) = 0, \text{ a.e. in } (0, T),$$

and therefore we may assume that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{t_0-\varepsilon}^{t_0+\varepsilon} (d\omega_s, \sum_{i=1}^N y_{i\eta}^* \alpha_{i\eta}(t)) = 0, \quad \forall \eta > 0. \quad (3.62)$$

By (3.61) and (3.62) we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_0^T (d\omega_s, \rho_\varepsilon) = 0.$$

By (3.59) and (3.60) we get

$$(\omega_a(t_0), y^*(t_0) - y) \geq 0, \quad \text{a.e. } t_0 \in (0, T). \quad (3.63)$$

Since y is arbitrary, (3.57) holds.

To conclude the proof it remains to be shown that $d\omega_s \in \mathcal{N}_{\mathcal{H}}(y^*)$, that is $\int_0^T (d\omega_s, y^* - x) \geq 0$, $\forall x \in \mathcal{H}$. Let \mathcal{O} be the support of the singular measure $d\omega_s$. Then for any $\varepsilon > 0$, there exists an open subset \mathcal{U} of $(0, T)$, s.t. $\mathcal{O} \subset \mathcal{U}$ and $m(\mathcal{U}) \leq \varepsilon$, where m is the lebesgue measure. Let $\rho \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \rho \leq 1$, $\rho = 1$ on \mathcal{O} and $\rho = 0$ on $(0, T) \setminus \mathcal{U}$. We set $y^\varepsilon = \rho x + (1 - \rho)y^*$, where $x \in \mathcal{H}$ is arbitrary. By (3.6)(or(3.8)), we have

$$\int_0^T (\omega_a(t), y^*(t) - y^\varepsilon(t)) dt + \int_0^T (d\omega_s, y^* - y^\varepsilon) \geq 0. \quad (3.64)$$

Since $y^* - y^\varepsilon = 0$ on $(0, T) \setminus \mathcal{U}$, we obtain that

$$\left| \int_0^T (\omega_a(t), y^*(t) - y^\varepsilon(t)) dt \right| \leq \int_{\mathcal{U}} |\omega_a| dt \leq \delta(\varepsilon),$$

where $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$. On the other hand, since $d\omega_s = 0$ on $(0, T) \setminus \mathcal{O}$ and $\rho = 1$ on \mathcal{O} , we see that

$$\int_0^T (d\omega_s, y^* - y^\varepsilon) = \int_0^T (d\omega_s, y^* - x).$$

It follows that $\int_0^T (d\omega_s, y^* - x) \geq -\delta(\varepsilon)$. Since ε is arbitrary, (3.58) holds. \sharp

4 Examples

In this section, we shall give some applications of the above results in some special cases of state constraints wherein Theorem 1 and Theorem 2 can be applied.

Example 1. Let K be the set $K = \{y \in H; \int_{\Omega} |y(x)|^2 dx \leq \rho^2\}$, then K is a closed convex set in H , since

$$\|\tilde{z}(t)\|_{C([0,T];H)} \leq C(\|B(y^*(t)) + D\tilde{u}(t) + f(t)\|_{L^2(0,T;H)})$$

so it is feasible to apply theorem 2 to get the necessary condition of the optimal control pair after checking whether condition (iv) is satisfied or not. The set K physically gives a constraint on the turbulence kinetic energy, which is usually bounded instead of infinite. In this case, the maximum principle can be described as following:

$$D^*p(t) \in \partial h(u^*(t)) \quad a.e. [0, T] \quad (4.1)$$

$$\begin{cases} p_1'(t) = \nu A p_1(t) + (B'(y^*(t))^*) p_1(t) + \mathcal{C}^* \mathcal{C}(y^*(t) - y^0(t)) + \omega_a(t), & a.e. \text{ in } (0, T) \\ p_1(T) = 0 \end{cases} \quad (4.2)$$

$$\begin{cases} p_2'(t) = \nu A p_2(t) + (B'(y^*(t))^*) p_2(t) + d\omega_s, & a.e. \text{ in } (0, T) \\ p_2(T) = 0 \end{cases} \quad (4.3)$$

Moreover,

$$\omega_a(t) \in N_K(y^*(t)) = \{\lambda(t)y^*(t); \lambda(t) \geq 0, a.e. \text{ in } (0, T)\} \quad (4.4)$$

Here p_1, p_2 is the decomposition of p , that is $p(t) = p_1(t) + p_2(t)$. Since $\omega_a(t) \in L^1(0, T; H)$, $d\omega_s$ is the singular part of the measure $d\omega$, we know that equation (4.2) has a strong solution $p_1 \in C([0, T]; H)$, while equation (4.3) has only a mild solution $p(t) = -\int_t^T U(s, t) dw_s$.

Example 2. Let K be the so called Enstrophy set

$$K = \{y \in V; \int_{\Omega} |\nabla \times y|^2 dx \leq \rho^2\}$$

where $\nabla \times y = \text{curl } y(x)$, and it is true that the norm $|\nabla \times y|$ is equivalent to the norm $\|y\|$ in the space V . In fluid mechanics, the enstrophy $\mathcal{E}(y) = \int_{\Omega} |\nabla \times y|^2 dx$ can be interpreted as another type of potential density. More precisely, the quantity directly related to the kinetic energy in the flow model that corresponds to dissipation effects in the fluid. It is particularly useful in the study of turbulent flows, and is often identified in the study of trusters, as well as the flame field. Enstrophy set gives a constraint on the vorticity of the fluid motion. Since

$$\|\tilde{z}(t)\|_{C([0,T];V)} \leq C(\|B(y^*(t)) + D\tilde{u}(t) + f(t)\|_{L^2(0,T;H)})$$

it is feasible to apply theorem 3 to get the necessary condition of the optimal control pair after checking whether condition (v) is satisfied or not. In this case, the maximum principle can be described by (4.1), (4.2) and (4.3). Moreover,

$$\omega_a(t) \in N_K(y^*(t)) = \{\lambda(t)Ay^*(t); \lambda(t) \geq 0, a.e. \text{ in } (0, T)\}$$

Example 3. Let K be the so called Helicity set,

$$K = \{y \in V; \int_{\Omega} \langle y, \text{curl } y \rangle^2 dx + \lambda \int_{\Omega} |\nabla y|^2 dx \leq \rho^2\}$$

where λ, ρ are positive constants. In fluid mechanics, helicity is the extent to which corkscrew-like motion occurs. If a parcel of fluid is moving, undergoing solid body motion rotating about an axis parallel to the direction of motion, it will have helicity. If the rotation is clockwise when viewed from ahead of the body, the helicity will be positive, if counterclockwise, it will be negative. Helicity is a useful concept in theoretical descriptions of turbulence. Formally, helicity is defined as

$$H = \int_{\Omega} \langle y, \text{curl } y \rangle dx$$

The helicity set plays an important role in fluid mechanics, and in particular, it is an invariant set of Euler's equation for incompressible fluids(See [4]). This set gives a constraint on the helicity and the smoothness of the velocity field. By the same argument as in Example 2, we know that it is feasible to apply theorem 3 to get the necessary condition of the optimal pair when the state constrained set is Helicity set, and in this case, the maximum principle can be described by (4.1), (4.2), (4.3). Moreover,

$$\omega_a(t) \in N_K(y^*(t)) = \{\lambda(t)(Ay^*(t) + \text{curl } y^*); \lambda(t) \geq 0, a.e. \text{ in } (0, T)\}$$

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